

Preferred attachment model of affiliation network

Mindaugas Bloznelis and Friedrich Götze

Vilnius University
LT-03225 Vilnius
Lithuania

Bielefeld University
D-33501 Bielefeld
Germany

Abstract

In an affiliation network vertices are linked to attributes and two vertices are declared adjacent whenever they share a common attribute. For example, two customers of an internet shop are called adjacent if they have bought the same or similar items. Assuming that each newly arrived customer is linked preferentially to already popular items we obtain a preferred attachment model of an evolving affiliation network. We show that the network has a scale-free property and establish the asymptotic degree distribution.

1 Introduction and results

A preferential attachment model of evolving network assumes that each newly arrived vertex is attached preferentially to already well connected sites, [2]. The preferential attachment principle is usually realised by setting the probability of a link between the new vertex v' and an old vertex v to be an increasing function of the degree of v (the number of neighbours of v). This scheme can be adapted to affiliation networks. In an affiliation network vertices are linked to attributes and two vertices are declared adjacent whenever they share a common attribute. For example two customers of an internet shop are called adjacent if they have bought the same or similar items. Here the preferred attachment principle means that a newly arrived customer is linked preferentially to already highly popular items, thus, further increasing their popularity. In the present study we show that a preferred attachment model of an affiliation network has a scale-free property and establish the asymptotic degree distribution.

Model. Given $\lambda > 0$ and integer $k > 0$, let $l \geq 0$ be an integer such that $\lambda \leq k + l$. Consider an internet library which contains w_1, \dots, w_l books/items at the beginning. Every book w_j is prescribed initial score $s(w_j) = 1$. On the first step new books w_{l+1}, \dots, w_{l+k} arrive to the library, each having initial score 1. Then the first customer v_1 visits the library and downloads books independently at random: a book w is chosen with probability $p_{1,s(w)} = \lambda s(w)(l + k)^{-1}$. Every book chosen by v_1 increases its score by one.

The collection of books of the library after n steps is denoted $W_n = \{w_1, \dots, w_{l+nk}\}$. On the $n + 1$ th step k new books arrive to the library, each having initial score 1. Then the customer v_{n+1} enters the library and downloads books of the library independently at random: a book w is downloaded with probability

$$p_{n+1,s(w)} = \lambda s(w)(l + (n + 1)k + n\lambda)^{-1}$$

proportional to the score $s(w)$ of w . Here $s(w) - 1$ is the number of vertices from $V_n = \{v_1, \dots, v_n\}$ that have downloaded the book w . Every book chosen by v_{n+1} increases its score by one.

We may interpret books as bins. Each newly arrived bin contains a single ball. A new customer v_{n+1} throws balls into bins $w_1, \dots, w_{l+(n+1)k}$ at random: each bin w receives a ball with probability $p_{n+1,s(w)}$ and independently of the other bins. The score $s(w)$ counts the (current) number of balls in the bin w . This number may increase with n . It measures the popularity (attractiveness) of the book w . Hence, popular books have higher chances to be chosen.

We call customers v_s and v_t adjacent if some book has been downloaded by both of them. We are interested in the graph G_n on the vertex set V_n defined by this adjacency relation.

Results. In the present note we address the question about the degree sequence of G_n . We shall show that for every $i = 0, 1, \dots$, the number of vertices $v \in V_n$ of G_n having degree $d(v) = i$ converges to a limit and identify this limit. Namely, we have as $n \rightarrow +\infty$

$$\begin{aligned} \frac{\#\{v \in V_n : d(v) = i\}}{n} &\rightarrow (1 + \alpha) \mathbf{E}\mathbb{I}_{\{Z \leq i, \Lambda \geq 1\}} \frac{\Gamma(i + 2\Lambda)}{\Gamma(i + 2\Lambda + \alpha + 2)} \frac{\Gamma(Z + 2\Lambda + \alpha + 1)}{\Gamma(Z + 2\Lambda)}, \quad i \geq 1, \\ \frac{\#\{v \in V_n : d(v) = 0\}}{n} &\rightarrow \mathbf{E}\mathbb{I}_{\{Z=0\}} \frac{1 + \alpha}{2\Lambda + 1 + \alpha}. \end{aligned} \quad (1)$$

Here $\alpha = k/\lambda$. Γ denotes Euler's Gamma function. Λ denotes a Poisson random variable with mean λ . Z is a compound Poisson random variable

$$Z = \sum_{i=1}^{\Lambda} T_i, \quad (2)$$

where T_1, T_2, \dots are independent random variables independent of Λ and having the same probability distribution

$$\mathbf{P}(T_1 = j) = x_{j+1}, \quad x_{j+1} = (1 + \alpha) \Gamma(2 + \alpha) \frac{\Gamma(j + 1)}{\Gamma(3 + \alpha + j)}, \quad j = 0, 1, 2, \dots \quad (3)$$

From (1) we find the tail behaviour of the limiting degree distribution. Let y_i denote the quantity on the right hand side of (1). We have as $i \rightarrow +\infty$

$$y_i \sim \lambda(1 + \alpha)^2 \Gamma(2 + \alpha) i^{-2-\alpha} \ln i. \quad (4)$$

Here and below we write $z_i \sim q_i$ whenever $z_i/q_i \rightarrow 1$ as $i \rightarrow +\infty$.

Numbers x_i have interesting interpretation. They are limits of the fractions of the number of books having score i :

$$\lim_{n \rightarrow +\infty} \frac{\#\{w \in W_n : s(w) = i\}}{nk} = x_i, \quad i = 1, 2, \dots \quad (5)$$

From the properties of Gamma function (formula (6.1.46) of [1]) we conclude that the sequence $\{x_i\}_{i \geq 1}$ obeys a power law with exponent $2 + \alpha$,

$$x_i \sim (1 + \alpha) \Gamma(2 + \alpha) i^{-2-\alpha} \quad \text{as } i \rightarrow +\infty. \quad (6)$$

Related work. Results of an empirical study of an evolving coauthorship network (an affiliation network, where authors are declared adjacent if they have a joint publication) are reported in [10]. The model considered in the present paper seems to be new. The idea of such a model has been suggested by Colin Cooper. The extra logarithmic factor in (4) indicates that the degree distribution of the preferred attachment affiliation model has a slightly heavier tail in comparison to that of the related 'usual' preferential attachment model, see [6], [7], [9]. On the other

hand, affiliation network models, where power law scores (6) are prescribed to items/attributes independently of the choices of vertices, have much heavier tails: the proportion of vertices of degree i scales as $i^{-1-\alpha}$ as $i \rightarrow +\infty$, see [3], [5]. An important property of real affiliation networks is that they admit a non-vanishing clustering coefficient, [11]. Clustering characteristics of the preferred attachment affiliation model will be considered elsewhere.

The paper is organized as follows. A heuristic argument explaining (1) and (5) is given in Section 2. A rigorous proof of (1), (4) and (5) is given in Section 3.

2 Heuristic

We start with explaining formula (5). Given $n \geq 1$ and $w \in W_n$, we denote by $s_n(w)$ the score of w after the n th step. By $X_i^{(n)}$ we denote the number of bins $w \in W_n$ of score $s_n(w) = i$. We put $X_1^{(0)} = l$ and $X_i^{(0)} = 0$, for $i \geq 2$.

Assume for a moment that for each i the ratios $X_i^{(n)}/(nk)$ converge to some limit, say \bar{x}_i , as $n \rightarrow +\infty$. So that for large n we have $X_i^{(n)} \approx \bar{x}_i nk$. Then from the relations describing approximate behaviour of the numbers $X_i^{(n)}$,

$$\begin{aligned} X_1^{(n+1)} &\approx (X_1^{(n)} + k)(1 - p_{n+1,1}), \\ X_2^{(n+1)} &\approx X_2^{(n)}(1 - p_{n+1,2}) + (X_1^{(n)} + k)p_{n+1,1}, \\ X_i^{(n+1)} &\approx X_i^{(n)}(1 - p_{n+1,i}) + X_{i-1}^{(n)}p_{n+1,i-1}, \quad i = 3, 4, \dots, \end{aligned}$$

we obtain, by neglecting $O(n^{-1})$ terms, the equations

$$\begin{aligned} \bar{x}_1(n+1)k &= (\bar{x}_1 nk + k) \left(1 - \frac{1}{n} \frac{1}{1+\alpha}\right), \\ \bar{x}_i(n+1)k &= \bar{x}_i nk \left(1 - \frac{1}{n} \frac{i}{1+\alpha}\right) + \bar{x}_{i-1} k \frac{i-1}{1+\alpha}, \quad i \geq 2. \end{aligned}$$

Solving these equations we arrive to the sequence $\{x_i\}_{i \geq 1}$ given by formula (3). We remark that $\{x_i\}_{i \geq 1}$ is a sequence of probabilities having a finite first moment. More precisely, we have

$$\sum_{i \geq 1} x_i = 1, \quad \sum_{i \geq 1} i x_i = 1 + \alpha^{-1}. \quad (7)$$

In particular, the common probability distribution of random variables T_i is well defined. We note that identities (7) are simple consequences of the well known properties of the Gamma function and hypergeometric series (formulas (6.1.46), (15.1.20) of [1]).

Next we explain (1). We call $w \in W_n$ and $v \in V_n$ related whenever w contains a ball produced by v . The number of balls produced by v is called the activity of v . A vertex $v \in V_n$ is called regular in G_n if every vertex adjacent to v in G_n shares with v a single bin. Introduce event $\mathcal{V}_{i,r} = \{v_{n+1} \text{ has activity } r, \text{ it has degree } i \text{ in } G_{n+1}, \text{ and it is a regular vertex of } G_{n+1}\}$ and let $q_{i,r}^{(n)}$ denote its probability. We observe that, given $X_1^{(n)}, X_2^{(n)}, \dots$, the conditional probability of the event $\mathcal{V}_{i,r}$ is

$$q_r^{(n)} \sum_{\substack{u_1+c+\dots+u_{i+1}=r, \\ 1u_2+2u_3+\dots+iu_{i+1}=i}} \frac{(X_1^{(n)} + k)_{u_1} (X_2^{(n)})_{u_2} \cdots (X_{i+1}^{(n)})_{u_{i+1}}}{(X_1^{(n)} + X_2^{(n)} + \cdots)_r} \frac{r!}{u_1! \cdots u_{i+1}!} + o(1). \quad (8)$$

Here we use notation $(x)_u = x(x-1)\cdots(x-u+1)$, u_s counts those bins $w \in W_{n+1}$ of score $s_n(w) = s$ that have received a ball from v_{n+1} , and $q_r^{(n)}$ is the conditional probability, given $X_1^{(n)}, X_2^{(n)}, \dots$, of the event that v_{n+1} has produced r balls. The remainder $o(1)$ accounts for the probability that v_{n+1} is not a regular vertex of G_{n+1} .

Now, using the approximations $X_i^{(n)} \approx x_i n$, $i \geq 1$, and identities (7) we, firstly, approximate the first fraction of (8) by $x_1^{u_1} \cdots x_{i+1}^{u_{i+1}}$ and, secondly, we approximate $q_r^{(n)}$ by the Poisson probability $e^{-\lambda} \lambda^r / r!$. We obtain that

$$q_{i,r}^{(n)} \approx e^{-\lambda} \frac{\lambda^r}{r!} \sum_{\substack{u_1 + \dots + u_{i+1} = r, \\ 1u_2 + 2u_3 + \dots + iu_{i+1} = i}} x_1^{u_1} x_2^{u_2} \cdots x_{i+1}^{u_{i+1}} \frac{r!}{u_1! \cdots u_{i+1}!} =: c_{i,r}.$$

Furthermore, we call $v \in V_n$ an $[i, r]$ vertex if its activity is r and its degree in G_n is $d(v) = i$. By $s_{i,r}(v)$ we denote the (current) number of balls contained in the bins related to an $[i, r]$ vertex v of G_n . We note that any regular $[i, r]$ vertex v of G_n has $s_{i,r}(v) = i + 2r =: s_{i,r}$. Moreover, the probability that v_{n+1} sends a ball to a bin related to such a vertex v is $s_{i,r} p_{n+1,1} + O(n^{-2})$.

Let $Y_i^{(n)}$ denote the number of regular vertices of G_n of degree $d(v) = i$, and let $Y_{i,r}^{(n)}$ denote the number of regular $[i, r]$ vertices of G_n . Assume for a moment that for each i and r the ratios $Y_i^{(n)} / n$ converge to some limit, say \bar{y}_i , and $Y_{i,r}^{(n)} / n$ converge to some limit, say $\bar{y}_{i,r}$, as $n \rightarrow +\infty$. So that for large n we have $Y_i^{(n)} \approx \bar{y}_i n$ and $Y_{i,r}^{(n)} \approx \bar{y}_{i,r} n$. Invoking these approximations in the relations describing approximate behaviour of numbers $Y_{i,r}^{(n)}$,

$$\begin{aligned} Y_{0,0}^{(n+1)} &\approx Y_{0,0}^{(n)} + q_{0,0}^{(n)}, \\ Y_{0,r}^{(n+1)} &\approx Y_{0,r}^{(n)} (1 - s_{0,r} p_{n+1,1}) + q_{0,r}^{(n)}, \quad r \geq 1, \\ Y_{i,r}^{(n+1)} &\approx Y_{i,r}^{(n)} (1 - s_{i,r} p_{n+1,1}) + Y_{i-1,r}^{(n)} s_{i-1,r} p_{n+1,1} + q_{i,r}^{(n)}, \quad i, r \geq 1. \end{aligned}$$

we obtain, by neglecting $O(n^{-1})$ terms and using the approximation $q_{i,r}^{(n)} \approx c_{i,r}$, the equations

$$\begin{aligned} \bar{y}_{0,0} &= c_{0,0}, \\ \bar{y}_{0,r} &= \frac{1 + \alpha}{1 + \alpha + 2r} c_{0,r}, \quad r \geq 1, \end{aligned} \tag{9}$$

$$\bar{y}_{i,r} = \frac{2r + i - 1}{1 + \alpha + 2r + i} \bar{y}_{i-1,r} + \frac{1 + \alpha}{1 + \alpha + 2r + i} c_{i,r}, \quad i, r \geq 1. \tag{10}$$

Solving these equations we arrive to the sequence $\{y_{0,0}, y_{i,r}, i \geq 0, r \geq 1\}$ given by the formulas

$$y_{0,0} = c_{0,0}, \tag{11}$$

$$y_{i,r} = (1 + \alpha) \sum_{j=0}^i \frac{(2r + i - 1)_{i-j}}{(1 + \alpha + 2r + i)_{i-j+1}} c_{j,r}. \tag{12}$$

Next we use the identity $c_{j,r} = \mathbf{P}(Z = j, \Lambda = r) = \mathbf{E} \mathbb{I}_{\{\Lambda=r\}} \mathbb{I}_{\{Z=j\}}$ and write (12) in the form

$$y_{i,r} = (1 + \alpha) \mathbf{E} \mathbb{I}_{\{\Lambda=r\}} \mathbb{I}_{\{Z \leq i\}} \frac{(2\Lambda + i - 1)_{i-Z}}{(1 + \alpha + 2\Lambda + i)_{i-Z+1}}.$$

Hence we obtain, for $i \geq 1$,

$$\begin{aligned}
y_i &= \sum_{r \geq 1} y_{i,r} \\
&= (1 + \alpha) \mathbf{E} \mathbb{I}_{\{\Lambda \geq 1\}} \mathbb{I}_{\{Z \leq i\}} \frac{(i + 2\Lambda - 1)_{i-Z}}{(i + 2\Lambda + \alpha + 1)_{i-Z+1}} \\
&= (1 + \alpha) \mathbf{E} \mathbb{I}_{\{\Lambda \geq 1\}} \mathbb{I}_{\{Z \leq i\}} \frac{\Gamma(i + 2\Lambda)}{\Gamma(i + 2\Lambda + \alpha + 2)} \frac{\Gamma(Z + 2\Lambda + \alpha + 1)}{\Gamma(Z + 2\Lambda)},
\end{aligned}$$

and

$$y_0 = \sum_{r \geq 0} y_{0,r} = \mathbf{P}(\Lambda = 0) + \mathbf{E} \mathbb{I}_{\{\Lambda \geq 1\}} \mathbb{I}_{\{Z=0\}} \frac{1 + \alpha}{2\Lambda + \alpha + 1} = \mathbf{E} \mathbb{I}_{\{Z=0\}} \frac{1 + \alpha}{2\Lambda + \alpha + 1}.$$

We remark that these identities imply (1), because for every $i \geq 0$, the number of non regular vertices of G_n of degree i can be shown to be negligible.

3 Appendix

Let $\tilde{Y}_{i,r}^{(n)}$ denote the number of non regular $[i, r]$ vertices of G_n .

Proof of (1), (4), (5). Let us prove (1). Let S_n denote the total number of balls in the network after the n -th step. A simple induction argument shows that $\mathbf{E} S_n = l + nk + n\lambda$. Let $\mathring{Y}_r^{(n)}$ denote the number of vertices $v \in V_n$ with activity at least r . We observe that for any $0 < \varepsilon < 1$

$$\sup_n \mathbf{P}(n^{-1} \mathring{Y}_r^{(n)} > \varepsilon) \rightarrow 0 \quad (13)$$

as $r \rightarrow \infty$. Indeed, vertices of V_n with activity at least r contribute at least $r \mathring{Y}_r^{(n)}$ balls to S_n . Hence, $\mathring{Y}_r^{(n)} \leq r^{-1} S_n$ and we obtain (13), by Markov's inequality. Now (1) follows from (13) and the fact that $n^{-1} \tilde{Y}_{i,r}^{(n)} \rightarrow 0$ and $n^{-1} Y_{i,r}^{(n)} \rightarrow y_{i,r}$ in probability as $n \rightarrow +\infty$ for $(i, r) = (0, 0)$ and $i \geq 0, r \geq 1$. This fact follows from Lemma 2: we have $n^{-1} \mathbf{E} \tilde{Y}_{i,r}^{(n)} \rightarrow 0$, $n^{-1} \mathbf{E} Y_{i,r}^{(n)} \rightarrow y_{i,r}$ and $\mathbf{Var}(n^{-1} Y_{i,r}^{(n)}) \rightarrow 0$.

Relation (5) follows from (15): we have $(nk)^{-1} \mathbf{E} X_i^{(n)} \rightarrow x_i$ and $\mathbf{Var}((nk)^{-1} X_i^{(n)}) \rightarrow 0$.

Let us prove (4). Since the Poisson random variable Λ is highly concentrated around its (finite) mean, we can approximate with a high probability for $i, z \rightarrow +\infty$

$$\frac{\Gamma(i + 2\Lambda)}{\Gamma(i + 2\Lambda + \alpha + 2)} \approx i^{-2-\alpha}, \quad \frac{\Gamma(z + 2\Lambda + \alpha + 1)}{\Gamma(z + 2\Lambda)} \approx z^{1+\alpha}.$$

Hence, we obtain $y_i \sim (1 + \alpha) \mathbf{E} Z^{1+\alpha} \mathbb{I}_{\{Z \leq i\}}$ as $i \rightarrow +\infty$. Next, to the randomly stopped sum Z of independent random variables T_i we apply the relation $\mathbf{P}(Z > t) \sim \mathbf{P}(T_1 > t) \mathbf{E} \Lambda$, [8]. We obtain $\mathbf{P}(Z > t) \sim \lambda \Gamma(2 + \alpha) t^{-1-\alpha}$. The latter relation implies $\mathbf{E} Z^{1+\alpha} \mathbb{I}_{\{Z \leq i\}} \sim \lambda(1 + \alpha) \Gamma(2 + \alpha) \ln i$ for $i \rightarrow +\infty$. We have arrived to (4). \square

The remaining part of the section contains auxiliary lemmas.

We write for short $p_{n+1,s} = p_s = s \varkappa_n$, where

$$\varkappa_n = p_{n+1,1} = \frac{1}{n} \frac{1}{1 + \alpha} \left(1 - \frac{1}{n} \frac{\alpha + \beta}{1 + \alpha + n^{-1}\alpha + n^{-1}\beta} \right), \quad \beta := \frac{l}{\lambda}. \quad (14)$$

Denote

$$\begin{aligned} x_i^{(n)} &= (nk)^{-1} \mathbf{E}X_i^{(n)}, & y_{i,r}^{(n)} &= n^{-1} \mathbf{E}Y_{i,r}^{(n)}, & \tilde{y}_{i,r}^{(n)} &= n^{-1} \mathbf{E}\tilde{Y}_{i,r}^{(n)}, \\ h_{i,j}^{(n)} &= (nk)^{-1} \left(\mathbf{E}X_i^{(n)} X_j^{(n)} - \mathbf{E}X_i^{(n)} \mathbf{E}X_j^{(n)} \right), & g_{i,j;r}^{(n)} &= n^{-2} \left(\mathbf{E}Y_{i,r}^{(n)} Y_{j,r}^{(n)} - \mathbf{E}Y_{i,r}^{(n)} \mathbf{E}Y_{j,r}^{(n)} \right). \end{aligned}$$

Lemma 1. For any $i, j \geq 1$ we have as $n \rightarrow +\infty$

$$x_i^{(n)} = x_i + O(n^{-1}), \quad (nk)^{-2} \mathbf{E}X_i^{(n)} X_j^{(n)} = x_i x_j + O(n^{-1}). \quad (15)$$

Moreover, the finite limits

$$h_{i,j} = \lim_n h_{i,j}^{(n)}, \quad i, j \geq 1, \quad (16)$$

exist and can be calculated using the recursive relations

$$h_{i,i} = \frac{2(i-1)h_{i,i-1} + ix_i + (i-1)x_{i-1}}{i + i + 1 + \alpha}, \quad (17)$$

$$h_{i,i+1} = \frac{(i-1)h_{i-1,i+1} + ih_{i,i} - ix_i}{i + (i+1) + 1 + \alpha}, \quad (18)$$

$$h_{i,r} = \frac{(i-1)h_{i-1,r} + (r-1)h_{i,r-1}}{i + r + 1 + \alpha}, \quad r \geq i + 2. \quad (19)$$

In particular, we have for every $i, j \geq 1$,

$$(nk)^{-2} \mathbf{E}X_i^{(n)} X_j^{(n)} = x_i^{(n)} x_j^{(n)} + h_{i,j}(nk)^{-1} + o(n^{-1}). \quad (20)$$

Here we use notation $x_0 \equiv 0$ and $h_{i,j} \equiv 0$, for $\min\{i, j\} = 0$.

Proof of Lemma 1. Let us prove the first relation of (15). The identities

$$\begin{aligned} \mathbf{E}X_1^{(n+1)} &= (1 - p_1) \mathbf{E}(X_1^{(n)} + k), \\ \mathbf{E}X_2^{(n+1)} &= (1 - p_2) \mathbf{E}X_2^{(n)} + p_1 \mathbf{E}(X_1^{(n)} + k), \\ \mathbf{E}X_i^{(n+1)} &= (1 - p_i) \mathbf{E}X_i^{(n)} + p_{i-1} \mathbf{E}X_{i-1}^{(n)}, \quad i \geq 1, \end{aligned}$$

imply

$$x_1^{(n+1)} = x_1^{(n)}(1 - n^{-1} - p_1) + n^{-1} + O(n^{-2}), \quad (21)$$

$$x_i^{(n+1)} = x_i^{(n)}(1 - n^{-1} - p_i) + x_{i-1}^{(n)} p_{i-1} + O(n^{-2}), \quad i \geq 1. \quad (22)$$

Relation (21) combined with Lemma 3 implies $x_1^{(n)} = x_1 + O(n^{-1})$. For $i \geq 2$ we proceed recursively: using the fact that $x_{i-1}^{(n)} = x_{i-1} + O(n^{-1})$ we conclude from (22) by Lemma 3 that $x_i^{(n)} = x_i + O(n^{-1})$.

Next, we observe that the second relation of (15) follows from (20). Furthermore, (20) follows from (17), (18) and (19). Hence we only need to prove (17), (18) and (19).

For convenience we write $h_{i,j}^{(n)} \equiv 0$, for $\min\{i, j\} = 0$. We also put $x_0^{(n)} \equiv 0$. Clearly, $h_{i,j}^{(n)} = h_{j,i}^{(n)}$ for $i, j \geq 0$.

Let us prove (17). A straightforward calculation shows that

$$\begin{aligned} h_{i,i}^{(n+1)} \frac{n+1}{n} &= h_{i,i}^{(n)}(1 - p_i)^2 + h_{i,i-1}^{(n)} 2(1 - p_i)p_{i-1} \\ &+ x_{i-1}^{(n)}(p_{i-1} - p_{i-1}^2) + x_i^{(n)}(p_i - p_i^2) \frac{n}{n+1} + O(n^{-2}), \end{aligned} \quad (23)$$

$$\begin{aligned} h_{i,i+1}^{(n+1)} \frac{n+1}{n} &= h_{i,i+1}^{(n)}(1 - p_i)(1 - p_{i+1}) + h_{i-1,i+1}^{(n)} p_{i-1}(1 - p_{i+1}) \\ &+ h_{i,i}^{(n)} p_i(1 - p_i) - x_i^{(n)} p_i(1 - p_i) + O(n^{-2}), \end{aligned} \quad (24)$$

and, for $r \geq 2 + i$,

$$\begin{aligned} h_{i,r}^{(n+1)} \frac{n+1}{n} &= h_{i,r}^{(n)}(1-p_i)(1-p_r) + h_{i-1,r}^{(n)} p_{i-1}(1-p_r) \\ &+ h_{i,r-1}^{(n)} p_{r-1}(1-p_i) + O(n^{-2}). \end{aligned} \quad (25)$$

We note that (23) and Lemma 3 imply that the sequence $\{h_{1,1}^{(n)}\}_{n \geq 1}$ converges to $h_{1,1}$ defined by (17). Furthermore, using the fact that (16) holds for $i = j = 1$ we obtain from (24) and Lemma 3 that $\{h_{1,2}^{(n)}\}_{n \geq 1}$ converges to $h_{1,2}$ defined by (18). Next, for $i = 1$ and $r = 3, 4, \dots$, we proceed recursively: using (25) and Lemma 3 we establish (16), with h_{ir} given by (19). In this way we prove the lemma for $i = 1$ and $r \geq i$.

The case $i = 2$, $r \geq i$ is treated similarly. For $i = r = 2$ we apply (23) and Lemma 3. For $i = 2$ and $r = 3$ we apply (24) and Lemma 3. Finally, for $i = 2$ and $r \geq i + 2$ we apply (19) and Lemma 3.

Next we proceed recursively and prove the lemma for $\{(i, r), r = i, r = i + 1, r = i + 2, \dots\}$, $i = 3, 4, \dots$. \square

Lemma 2. *Let $i, j = 0, 1, \dots$ and $r = 1, 2, \dots$. We have as $n \rightarrow +\infty$*

$$y_{i,r}^{(n)} \rightarrow y_{i,r}, \quad g_{i,j;r}^{(n)} \rightarrow 0, \quad \tilde{y}_{i,r}^{(n)} \rightarrow 0. \quad (26)$$

(26) remains valid for $i = j = r = 0$.

Proof of Lemma 2. For $i, j, r \geq 1$ we show that

$$\tilde{y}_{0,0}^{(n)} \equiv \tilde{y}_{0,r}^{(n)} \equiv \tilde{y}_{i,1}^{(n)}, \quad (27)$$

$$\tilde{y}_{i+1,r}^{(n+1)} \leq \tilde{y}_{i+1,r}^{(n)}(1-n^{-1}) + \tilde{y}_{i,r}^{(n)} s_{ir,r} \mathcal{K}_n + o(n^{-1}), \quad (28)$$

$$y_{0,0}^{(n+1)} = (1-n^{-1})y_{0,0}^{(n)} + n^{-1}c_{0,0} + o(n^{-1}), \quad (29)$$

$$y_{0,r}^{(n+1)} = (1-n^{-1} - s_{0,r} \mathcal{K}_n) y_{0,r}^{(n)} + n^{-1}c_{0,r} + o(n^{-1}), \quad (30)$$

$$y_{i,r}^{(n+1)} = (1-n^{-1} - s_{i,r} \mathcal{K}_n) y_{i,r}^{(n)} + s_{i-1,r} \mathcal{K}_n y_{i-1,r}^{(n)} + n^{-1}c_{i,r} + o(n^{-1}), \quad (31)$$

and

$$g_{0,0;0}^{(n+1)} = (1-2n^{-1})g_{0,0;0}^{(n)} + o(n^{-1}), \quad (32)$$

$$g_{0,0;r}^{(n+1)} = (1-2n^{-1} - 2s_{0,r} \mathcal{K}_n)g_{0,0;r}^{(n)} + o(n^{-1}), \quad (33)$$

$$g_{0,j;r}^{(n+1)} = (1-2n^{-1} - (s_{j,r} + s_{0,r}) \mathcal{K}_n)g_{0,j;r}^{(n)} + s_{j-1,r} \mathcal{K}_n g_{0,j-1;r}^{(n)} + o(n^{-1}), \quad (34)$$

$$\begin{aligned} g_{i,j;r}^{(n+1)} &= (1-2n^{-1} - (s_{i,r} + s_{j,r}) \mathcal{K}_n)g_{i,j;r}^{(n)} + s_{i-1,r} \mathcal{K}_n g_{i-1,j;r}^{(n)} + s_{j-1,r} \mathcal{K}_n g_{i,j-1;r}^{(n)} \\ &+ o(n^{-1}). \end{aligned} \quad (35)$$

The proof of (27)-(35) is technical. We refer the reader to the extended version of the paper [4] for details. Here we prove that (27)-(35) imply (26).

Let us prove the third relation of (26). For $i = 0$, and for $r = 0, 1$ the relation follows from (27). Next, for any fixed $r \geq 2$ we proceed recursively: from (28) combined with the fact that $\tilde{y}_{i,r}^{(n)} \rightarrow 0$ we conclude by Lemma 3 that $\tilde{y}_{i+1,r}^{(n)} \rightarrow 0$.

Let us prove the first and second relation of (26). Firstly, combining (29) (respectively (32)) with Lemma 3 we obtain the first (respectively second) relation of (26), for $i = j = r = 0$. Secondly, combining (30) (respectively (33)) with Lemma 3 we obtain the first (respectively second) relation of (26), for $i = j = 0, r \geq 1$.

Now we prove the first relation of (26) for $i \geq 1$ and $r \geq 1$. We fix r and proceed recursively: from the fact that $y_{i-1,r}^{(n)} \rightarrow y_{i-1,r}$ and relation (31) we conclude by Lemma 3 that $y_{i,r}^{(n)} \rightarrow y_{i,r}$. Next, we prove the second relation of (26) for $r \geq 1$ and $i + j \geq 1$. We fix r and proceed recursively in i and j .

For $i = 0$ and $j \geq 1$ we proceed as follows: from the fact that $g_{0,j-1;r}^{(n)} \rightarrow 0$ and relation (34) we conclude by Lemma 3 that $g_{0,j;r}^{(n)} \rightarrow 0$. In this way we prove the second relation of (26) for (i, j) such that $i = 0$ and $j \geq 1$.

Now, consider indices $i = 1$ and $j \geq 1$. From the fact that $g_{1,j-1;r}^{(n)} \rightarrow 0$ and relation (34) we conclude by Lemma 3 that $g_{1,j;r}^{(n)} \rightarrow 0$. In this way we prove the second relation of (26) for (i, j) such that $i = 1$ and $j \geq 2$.

Proceeding similarly we establish the second relation of (26) for $\{(i, i), (i, i + 1), (i, i + 2), \dots\}$, $i = 2, 3, \dots$. \square

Lemma 3. *Let $b, h \in \mathbb{R}$. Let $\{b_n\}_{n \geq 1}$ be a real sequence converging to b and assume that the series $\sum_{n \geq 1} n^{-1}|b_n - b|$ converges. Let $\{h_n\}_{n \geq 1}$ be a real sequence converging to h . Let $\{a_n\}_{n \geq 1}$ be a real sequence satisfying the recurrence relation*

$$a_{n+1} = a_n(1 - n^{-1}b_n) + n^{-1}h_n, \quad n \geq 1. \quad (36)$$

For $b > 0$ we have $a_n \rightarrow hb^{-1}$. Suppose, in addition, that $b_n - b = O(n^{-1})$, $h_n - h = O(n^{-1})$. Then for $b \neq 1$ we have $a_n - hb^{-1} = O(n^{-1 \wedge b})$, and for $b = 1$ we have $a_n - hb^{-1} = O(n^{-1} \ln n)$. Let $\tilde{b} \geq 0$. Let $\{\tilde{a}_n\}_{n \geq 1}$, $\{\tilde{b}_n\}_{n \geq 1}$, $\{\tilde{h}_n\}_{n \geq 1}$ be non negative sequences such that $\tilde{b}_n \rightarrow \tilde{b}$, $\tilde{h}_n \rightarrow 0$ and $\{\tilde{a}_n\}_{n \geq 1}$ satisfies the inequality

$$\tilde{a}_{n+1} \leq \tilde{a}_n(1 - n^{-1}\tilde{b}_n) + n^{-1}\tilde{h}_n, \quad n \geq 1.$$

Assume that the series $\sum_{n \geq 1} n^{-1}|\tilde{b}_n - \tilde{b}|$ converges. Then $\{\tilde{a}_n\}_{n \geq 1}$ converges to 0.

The proof is straightforward, see [4] for details.

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